Quasi-stationary distribution for branching processes with competition

Pei-Sen Li

Based on the joint work with Jian Wang and Xiaowen Zhou.

The 18th Workshop on Markov Processes and Related Topics

Outline

- Definition of Quasi-Stationary Distribution.
- Classical results for branching processes (without competition).
- Our results for branching processes with competition.
- Strategy of the proof.

Quasi-stationary distribution (QSD)

Consider a Markov process $(X_t)_{t\geq 0}$ taking values in E $(=\mathbb{N}$ or $=\mathbb{R}_+)$. Assume that $X_t = 0$ for $t \geq T_0$, where $T_0 = \inf\{t \geq 0 : X_t = 0\}$ is the extinction time.

Definition

We call a probability measure μ a quasi-stationary distribution if for all $t \geq 0$,

$$\mu(\cdot) = \mathbb{P}_\mu(X_t \in \cdot | T_0 > t).$$

- If μ is a QSD, then for some initial distribution α,

$$\lim_{t\to\infty} \mathbb{P}_\alpha(X_t \in \cdot | T_0 > t) = \mu.$$

- Question: existence and uniqueness of QSD; domain of attraction; exponential convergence.

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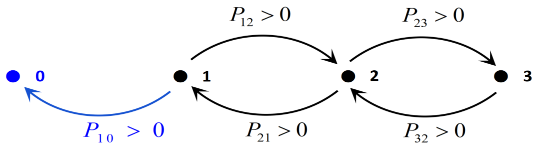

Let Q be the restriction of P on $\{1,2,3\}$. Consider the eigenequation

$$\mu Q = \alpha \mu,$$

where $\alpha \in (0,1)$ is the largest eigenvalue.

- There is a unique positive eigenvector μ solves the equation.
- The associated eigenvectors of any other eigenvalue cannot be positive.
- Summing up both sides, we see $\mathbb{P}_\mu(T_0 > 1) = \alpha$.
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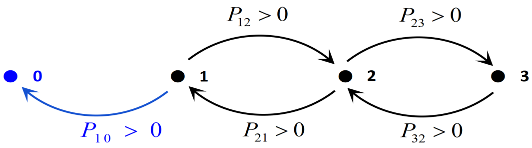

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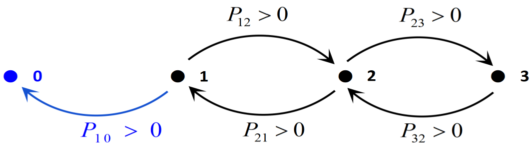

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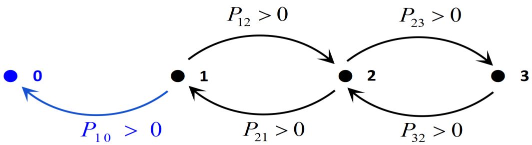

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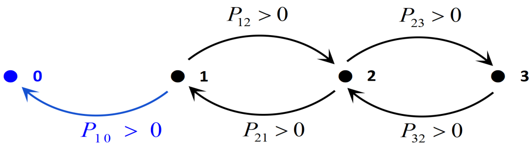

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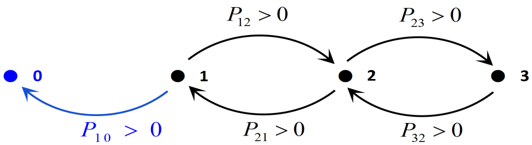

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QSDs for branching process

Yaglom (1947) first studied the QSDs for Galton-waston branching process:

$$X_n = \sum_{i=1}^{X_{n-1}} \xi_{n,i}, \quad n \geq 1.$$

where $(\xi_{n,i})$ are i.i.d. random variables in $\{0, 1, 2, \dots\}$. Let $\rho = \mathbb{E}(\xi_{n,i})$.

- If $\rho \geq 1$, then there is no QSD.
- If $\rho < 1$, then there are infinitely many QSDs.
  - Any $\alpha \in [\rho, 1)$ is a eigenvalue corresponding to a eigenvector μ_α, which is a QSD after normalisation.
  - Eigenvalue ρ corresponds to Yaglom's limit (1947): For any $x > 0$,

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Continuous-state branching processes

Consider a sequence of Galton–Watson branching processes

$$\{X_n^{(k)} : n \geq 0\}, k = 1, 2, \ldots.$$

A continuous-state branching process (CB-process) arises as the scaling limit

$$X_t = \lim_{k \to \infty} \frac{1}{k} X_{\lfloor kt \rfloor}^{(k)}.$$

The transition semigroup $(Q_t)_{t \geq 0}$ of (X_t) is defined by

$$\int_{\mathbb{R}_+} \mathrm{e}^{-\lambda y} Q_t(x, \mathrm{d}y) = \mathrm{e}^{-x v_t(\lambda)}, \quad \frac{\partial}{\partial t} v_t(\lambda) = -\Psi(v_t(\lambda)),$$

where the *branching mechanism* Ψ is given by

$$\Psi(\lambda) = b\lambda + c\lambda^2 + \int_0^\infty \left(\mathrm{e}^{-\lambda z} - 1 + \lambda z 1\!\!1_{\{z \leq 1\}}\right) m(\mathrm{d}z), \quad \lambda \geq 0.$$

- A CB-process with branching mechnism $\Psi(\lambda) = b\lambda + c\lambda^2$ is called *Feller branching diffusion*, which solves the SDE

$$Y_t = Y_0 - \int_0^t bY_{s-}\mathrm{d}s + \int_0^t \sqrt{2cY_{s-}}\mathrm{d}B_s,$$

where $(B_s)_{s\geq 0}$ is a Brownian motion.

- A CB-process with general branching mechanism Ψ can be constructed as the unique solution to the equation with jumps (Dawson–Li, AOP, 2006):

$$\begin{aligned} Y_t &= Y_0 - \int_0^t bY_{s-}\mathrm{d}s + \int_0^t \sqrt{2cY_{s-}}\mathrm{d}B_s + \int_0^t \int_0^1 \int_0^{Y_{s-}} z\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &\quad + \int_0^t \int_1^\infty \int_0^{Y_{s-}} zM(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned}$$

where $M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ is a Poisson random measure with intensity $\mathrm{d}sm(\mathrm{d}z)\mathrm{d}u$ and $\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ is the compensated measure.

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Condition I (Grey's condition)

There exists a constant $\theta > 0$ such that

$$\Psi(\lambda) > 0 \text{ for } \lambda > \theta \text{ and } \int_\theta^\infty \frac{\mathrm{d}\lambda}{\Psi(\lambda)} < \infty.$$

Let $\rho = \Psi'(0)$.

Lambert (EJP, 2007):

- If $\rho \le 0$ or Condition I is not satisfied, then there is no QSD.
- If $\rho > 0$ and Condition I is satisfied, then there are infinitely many QSDs.

Super processes: Liu-Ren-Song-Sun (SPA, 2021);

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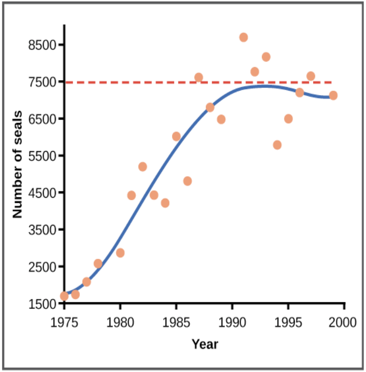
The proofs for CB-process essentially rely on the *branching property:*

$$Q_t(x,\cdot) * Q_t(y,\cdot) = Q_t(x+y,\cdot),$$

which means that different individuals act independently with each other.

"Population growth of harbor seals in Washington State"



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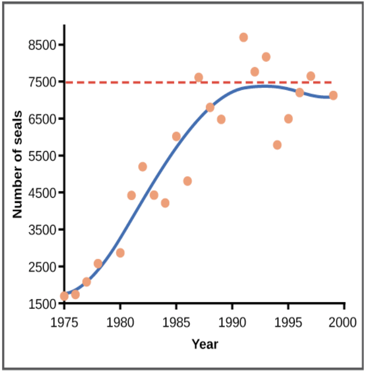

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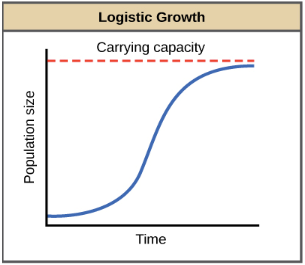
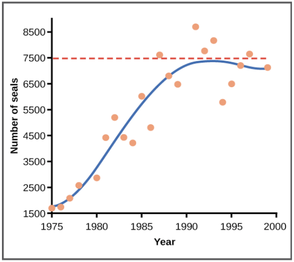
A deterministic model

A logistic growth model can be constructed as the solution to:

$$Y_t = Y_0 - \int_0^t (bY_s + \beta Y_s^2)\mathrm{d}s,$$

where $-b > 0$ and $\beta > 0$. The blue term describes competition between each pair of individuals.





Branching processes with competition

- Lambert (AAP, 2005) introduced the logistic branching process as the random time change of an OU-process.
- The process solves the SDE:

$$\begin{aligned} Y_t \ =& \ Y_0 - \int_0^t bY_{s-}\mathrm{d}s + \int_0^t \sqrt{2cY_{s-}}\mathrm{d}B_s - \int_0^t \beta Y_s^2 \mathrm{d}s \\ &+ \int_0^t \int_0^1 \int_0^{Y_{s-}} z\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_1^\infty \int_0^{Y_{s-}} zM(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned}$$

- The process is more natural and realistic. However, many tools fail to apply.

Berestycki-Fittipaldi-Fontbona (PTRF, 2018) introduced the general competition model :

$$\begin{aligned} Y_t &= Y_0 - \int_0^t bY_{s-}\mathrm{d}s + \int_0^t \sqrt{2cY_{s-}}\mathrm{d}B_s - \int_0^t g(Y_s)\mathrm{d}s \\ &\quad + \int_0^t \int_0^1 \int_0^{Y_{s-}} z\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_1^\infty \int_0^{Y_{s-}} zM(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned}$$

where the competition mechanism g is a nondecreasing function satisfying $g(0) = 0$.

- When $g(x) \equiv 0$, it reduces to a CB-process.
- When $g(x) = \beta x^2$ ($\beta > 0$), it reduces to the logistic model.
- They solved the question on genealogy asked by Lambert.
- For more general models (nonliner branching processes):
  - ◇ **Ergodicity:** L.-Wang (EJP, 2020), L.-Li-Wang-Zhou (AIHP, 2023+)
  - ◇ **Boundary behaviour:** Palau and Pardo (SPA, 2017), L.-Yang-Zhou (AAP, 2019), Ma-Yang-Zhou (ECP, 2021), Xiong-Ren-Yang-Zhou (S-PA, 2022), Marguet and Smadi (EJP, 2021).

Diffusive situation

Some progress on the uniqueness of the QSD has been made by Cattiaux et al. (AOP, 2009) for the diffusion process (a special CB-process with competition):

$$Y_t = Y_0 - \int_0^t bY_{s-}\mathrm{d}s + \int_0^t \sqrt{2cY_{s-}}\mathrm{d}B_s - \int_0^t g(Y_s)\mathrm{d}s.$$

Theorem (Cattiaux et al., AOP, 2009)

Suppose that

$$\int_1^\infty \frac{\mathrm{d}x}{g(x)} < \infty.$$

Then there is only one QSD, and this distribution attracts all initial distributions.

- The diffusion process has a symmetric measure μ. Their proof is based on the spectral theory of the space $L^2(\mu)$.
- A general CB-process with competition only has positive jumps, so there is no symmetric measure. The treatment of the process requires new tools.

Main result

Condition II

$$\int_1^\infty \frac{\mathrm{d}x}{g(x)} < \infty \qquad \text{and} \qquad \limsup_{x\to\infty} x \int_{x+1}^\infty \frac{1}{g(y)} \int_{y-x}^\infty m(\mathrm{d}z) < \infty.$$

Theorem (L.-Wang-Zhou 2023+)

Suppose that Condition I, II are satisfied. Then there is a unique quasi-stationary distribution μ. Moreover, there exist $\lambda > 0$ such that for any initial distribution ν on $(0, \infty)$ and $t > 0$,

$$||\mathbb{P}_\nu(\cdot|T_0 > t) - \mu(\cdot)||_{\mathrm{Var}} \leq C_\nu \mathrm{e}^{-\lambda t}.$$

- Intuitively speaking, the process has the unique QSD if both fluctuation and competition are strong enough.

Main result

Examples:

- Condition I is satisfied if $c > 0$ or $m(\mathrm{d}z) \geq \mathbb{1}_{\{0<z\leq 1\}} c_0 z^{-1-\alpha} \mathrm{d}z$ for some $\alpha \in (1,2)$ and $c_0 > 0$. (The diffusion term is not vanishing or there are sufficiently more small jumps.)
- Condition II is satisfied if there exist $c_0 > 0$ and $\beta > 1$ such that

$$\mathbb{1}_{\{z>1\}} m(\mathrm{d}z) \leq c_0 z^{-1-\alpha} \mathrm{d}z$$

and

$$\begin{cases} \lim_{x\to\infty} \frac{g(x)}{x^\beta} = \infty, & \alpha \in [1,2), \\ \lim_{x\to\infty} \frac{g(x)}{x^{2-\alpha}} = \infty, & \alpha \in (0,1). \end{cases}$$

(The number of big jumps is suitably controlled and the competition is sufficiently strong.)

Strategy of the proof

The proof is based on recent results of Guillin-Nectoux-Wu (2020+; PTRF, 2023). We need to verify:

(C1) Strong Feller property: For any $t > 0$ and $f \in b\mathscr{B}$, $x \mapsto P_t f(x)$ is continuous on $[0, \infty)$.

(C2) Trajectory Feller property: For any $T > 0$, $x \mapsto \mathbb{P}_x((Y_t)_{t\in[0,T]} \in \cdot)$ is continuous in the sense of weak convergence.

(C3) Weak Feller property: The killed transition semigroup is weakly Feller.

(C4) Irreducibility: For all $x, t > 0$ and open set $O \subset (0, \infty)$, $P_t(x, O) > 0$.

(C5): There exist a function $W \in C_b^2[0, \infty)$ such that $W(x) \geq 1$ for all $x \in [0, \infty)$, two sequences of positive constants $r_n \to \infty$ and b_n, and an increasing sequence of compact subsets $K_n \subset [0, \infty)$ such that

$$-LW(x) \geq r_n W(x) - b_n \mathbb{1}_{K_n}(x), \quad x \in [0, \infty).$$

Thank you for your attention!